

# Killing-Yano Forms of a Class of Spherically Symmetric Space-Times II: A Unified Generation of Higher Forms

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## Abstract

Killing-Yano (KY) two and three forms of a class of spherically symmetric space-times that includes the well-known Minkowski, Schwarzschild, Reissner-Nordström, Robertson-Walker and six different forms of de Sitter space-times as special cases are derived in a unified and exhaustive manner. It is directly proved that while the Schwarzschild and Reissner-Nordström space-times do not accept any KY 3-form and they accept only one 2-form, the Robertson-Walker space-time admits four KY 2-forms and only one KY 3-form. Maximal number of KY-forms are obtained for Minkowski and all known forms of de Sitter space-times. Complete lists comprising explicit expressions of KY-forms are given.

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## I. INTRODUCTION

In the previous study [1] (henceforth referred to as I), we developed a constructive method which provided a unified generation of all Killing vector fields for a class of four dimensional ( $4D$ ) spherically symmetric space-times. As the sequel of I, in the present paper we shall investigate the explicit forms of KY two and three forms by solving the KY-equations

$$\nabla_{X_a}\omega_{(p)} = \frac{1}{p+1}i_{X_a}d\omega_{(p)} , \quad (1)$$

for this class of space-time metrics in the  $p = 2$  and  $p = 3$  cases.

The underlying base manifold is supposed to be a  $4D$  pseudo-Riemannian manifold endowed with the metric  $g$  having the Lorentzian signature  $(-+++)$  such that

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 .$$

in a local orthonormal co-frame basis  $\{e^a\}$ . This metric can be parameterized by the (metric) characterizing functions  $T = \exp(\lambda(t))$  and  $H_j = H_j(r)$  by choosing

$$\begin{aligned} e^0 &= H_0 dt , & e^1 &= T H_1 dr , \\ e^2 &= T H_2 d\theta , & e^3 &= T H_2 \sin \theta d\varphi . \end{aligned}$$

The corresponding orthonormal vector bases will be denoted by  $X_a$ . The torsion-free connection 1-forms for this class of metrics and covariant derivatives of basis elements required for explicit calculations can be found in I. We shall mainly use the notation of I, which is in accordance with [2].

As they span the kernel of the linear operator  $\nabla_X - (p+1)^{-1}i_X d$ , the space  $Y_p$  of all KY  $p$ -forms constitute a linear space [3, 4]. Any function is a KY 0-form,  $\omega_{(1)}$  is the dual of a Killing vector field and  $\omega_{(n)}$  is a constant (parallel), that is, it is a constant multiple  $\omega_{(n)} = az$  of the volume form, say  $z = e^{0123}$  for  $n = 4$ . Therefore while  $Y_0$  is infinite dimensional,  $Y_n$  is one dimensional. By a straightforward extension of the argument for determining the maximum number of Killing vectors, the upper bound for the dimension of  $Y_p$ 's can now be established for each  $p$  [5, 6]. For this purpose, we should first note that equation (1) suggests an equivalent definition: a  $p$ -form is a KY  $p$ -form if and only if its symmetrized covariant derivatives vanish, that is, if and only if

$$i_X \nabla_Y \omega_{(p)} + i_Y \nabla_X \omega_{(p)} = 0 ,$$

is satisfied for all pair of the vector fields  $X$  and  $Y$ . This is also equivalent to  $i_X \nabla_X \omega_{(p)} = 0$  and in particular every KY-form is divergent free, or equivalently co-closed, that is,  $\delta \omega_{(p)} = -i_{X^a} \nabla_{X_a} \omega_{(p)} = 0$ . These statements imply that the covariant derivatives of KY forms are also totally anti-symmetric, with respect to the additional tensorial index. Hence the maximum numbers of linearly independent components and their first covariant derivatives are  $C(n, p)$  and  $C(n, p + 1)$  respectively, where  $C(n, p)$  stands for the binomial numbers.

On the other hand “the second covariant derivatives”, that is, the Hessian of KY forms can be written, in terms of the curvature 2-forms  $R_a^c$  as

$$(\nabla_{X_a} \nabla_{X_b} - \nabla_{\nabla_{X_a} X_b}) \omega_{(p)} = \frac{1}{p} i_{X_b} (R_a^c \wedge i_{X_c} \omega_{(p)}) .$$

These imply that the value of any component of a KY-form at any point is entirely determined by the value of its first covariant derivative and by the value of the component itself at the same point. As a result the upper bounds for the numbers of linearly independent KY-forms is determined by the sum  $C(n, p) + C(n, p + 1) = C(n + 1, p + 1)$ . In particular for  $n = 4$  the upper bounds are 10, 10 and 5, respectively, for  $p = 1, p = 2$  and  $p = 3$ . These bounds are attained for the space-times of constant curvature.

The rest of the paper is structured in two main parts. In the next section, the defining equations for KY 2-forms and their integrability conditions are obtained. By the integrability conditions, the set of solutions naturally breaks up into cases and subcases that are considered in the rest of first part consisting five sections. The second part is entirely devoted to determination of KY 3-forms. The final Section VIII presents a summary of the results, and we briefly point out how to calculate the associated Killing tensors and related linear and quadratic first integrals for the considered class of space-times.

## II. KY 2-FORMS: DEFINING EQUATIONS, INTEGRABILITY CONDITIONS

For a 2-form

$$\omega_{(2)} = \alpha e^{01} + \beta e^{02} + \gamma e^{03} + \delta e^{12} + \epsilon e^{13} + \mu e^{23} ,$$

there are two types of equations that result from KY-equation (1) in the case of  $p = 2$ . The first consists of twelve relatively simple equations resulting from the left hand side of the KY equations. Two of them are  $\alpha_t = 0 = \alpha_r$ , which imply that  $\alpha = \alpha(\theta, \varphi)$ , and the other

ten equations are as follows:

$$\begin{aligned}
\beta_t &= \frac{H'_0}{TH_1}\delta, & \beta_\theta &= -\frac{H'_2}{H_1}\alpha, \\
\delta_r &= \dot{T}\frac{H_1}{H_0}\beta, & \delta_\theta &= -\dot{T}\frac{H_2}{H_0}\alpha, \\
\gamma_t &= \frac{H'_0}{TH_1}\epsilon, & \gamma_\varphi &= -\frac{H'_2}{H_1}\sin\theta\alpha - \cos\theta\beta, \\
\epsilon_r &= \dot{T}\frac{H_1}{H_0}\gamma, & \epsilon_\varphi &= -\dot{T}\frac{H_2}{H_0}\sin\theta\alpha - \cos\theta\delta, \\
\mu_\theta &= \dot{T}\frac{H_2}{H_0}\gamma - \frac{H'_1}{H_1}\epsilon, & \mu_\varphi &= -\dot{T}\frac{H_2}{H_0}\sin\theta\beta + \frac{H'_2}{H_1}\sin\theta\delta.
\end{aligned} \tag{2}$$

The second type also consists of twelve equations, but four of them are just a sum of two other equations of the same type. Hence, there are eight independent additional equations, six of which can be put in the following forms:

$$\begin{aligned}
\alpha_\theta &= -\frac{H_2^2}{H_1}\partial_r\frac{\beta}{H_2}, & \alpha_\varphi &= -\frac{H_2^2}{H_1}\sin\theta\partial_r\frac{\gamma}{H_2}, \\
\partial_r\frac{\beta}{H_0} &= -T^2\frac{H_1}{H_0^2}\partial_t\frac{\delta}{T}, & \gamma_\theta &= -T^2\frac{H_2}{H_0}\partial_t\frac{\mu}{T}, \\
\partial_r\frac{\gamma}{H_0} &= -T^2\frac{H_1}{H_0^2}\partial_t\frac{\epsilon}{T}, & \epsilon_\theta &= -\frac{H_2^2}{H_1}\partial_r\frac{\mu}{H_2}.
\end{aligned} \tag{3}$$

The last two equations (presented below) do not depend on the metric coefficient functions, and therefore much of the essence of spherical symmetry is encoded in them:

$$\beta_\varphi = -\sin^2\theta\partial_\theta\frac{\gamma}{\sin\theta}, \quad \delta_\varphi = -\sin^2\theta\partial_\theta\frac{\epsilon}{\sin\theta}. \tag{4}$$

Before solving these eighteen equations, there are some integrability conditions that must be met. Two of them, which enable us to find the solutions in a systematic way, are

$$\dot{T}H'_0\alpha = 0, \tag{5}$$

$$[\rho(r) + \varrho(t)]\alpha = 0, \tag{6}$$

where

$$\rho = \frac{H_0^3}{H_1H_2}\left(\frac{H'_2}{H_0H_1}\right)', \quad \varrho = T\ddot{T} - \dot{T}^2 = T^2\partial_t\left(\frac{\dot{T}}{T}\right). \tag{7}$$

Condition (5) separately follows from each of

$$\beta_{t\theta} = \beta_{\theta t}, \quad \gamma_{t\varphi} = \gamma_{\varphi t}, \quad \delta_{r\theta} = \delta_{\theta r}, \quad \epsilon_{r\varphi} = \epsilon_{\varphi r},$$

and condition (6) can be checked from  $\delta_{t\theta} = \delta_{\theta t}$  or  $\epsilon_{\varphi t} = \epsilon_{t\varphi}$ . The integrability conditions imply that solutions must be investigated in two classes: (A)  $\alpha \neq 0$  and (B)  $\alpha = 0$ , such that the first consists of three important subclasses characterized by

$$(i) H'_0 = 0, \quad \dot{T} \neq 0, \quad (ii) H'_0 \neq 0, \quad \dot{T} = 0, \quad (iii) H'_0 = 0 = \dot{T}.$$

In fact there are  $6 \times 6$  integrability conditions, some of which are trivially satisfied and the remaining ones will be considered in the following classes. We should also note that whenever  $T$  or  $H_k$ 's are constant, they must be considered to be nonzero to keep the metric non-degenerate.

In the next three sections, we shall consider the first (A) case in which  $\alpha$  is different from zero. At the outset of these sections, we should note the following two equations for  $\alpha$ :

$$\alpha_{\theta\theta} + \ell_1 \alpha = 0, \quad (8)$$

$$\alpha_{\varphi\varphi} + \ell_1 \sin^2 \theta \alpha + \sin \theta \cos \theta \alpha_\theta = 0, \quad (9)$$

which do not change in the following subcases. These are obtained by differentiating the first two equations of (3) and by making use of the equation for  $\beta_\theta$  and  $\gamma_\varphi$  from (2). Here the constant  $\ell_1$  is defined, in terms of  $P = H'_2/H_1 H_2$ , as follows:

$$\ell_1 = -\frac{H_2^2}{H_1} P', \quad (10)$$

We should finally note that, as there are no equations involving the first power of  $\mu$ ,

$$\omega_0 = T H_2 e^{23}$$

is a solution of the KY-equation without any constraint. This observation means that all the space-times within the considered class of metrics admit of at least one KY 2-form. This fact was also observed for the static space-times in [7, 8]. In fact, it turns out that  $\omega_0$  is the only KY 2-form admitted by two important cases, the Schwarzschild and the Reissner-Nordström space-times, which do not accept any KY 3-forms.

### III. KY 2-FORMS FOR $H'_0 = 0, \dot{T} \neq 0, \alpha \neq 0$

In the case of nonzero  $\alpha$  and  $H'_0 = 0$ , two equations of (2) show that  $\beta$  and  $\gamma$  are time independent and the integrability conditions (5-7) require that  $\ell$ , defined in terms of  $P$  by

$$\frac{H_0^2}{H_1 H_2} (H_2 P)' = -\ell = T^2 \partial_t \left( \frac{\dot{T}}{T} \right), \quad (11)$$

must be a constant. By multiplying both sides of the first equality by  $H_2 H_2'$  it can easily be integrated to write

$$P^2 = \frac{\ell_1}{H_2^2} - \frac{\ell}{H_0^2} , \quad (12)$$

where  $\ell_1$  is taken as an integration constant, since the defining relation (10) of  $\ell_1$  is implied by the condition (12).

Then, in terms of

$$\begin{aligned} \mathcal{B} &= \dot{T} \frac{H_2}{H_0} \beta , & \mathcal{D} &= H_2 P \delta , \\ \mathcal{G} &= \dot{T} \frac{H_2}{H_0} \gamma , & \mathcal{E} &= H_2 P \epsilon , \end{aligned}$$

we define

$$x = \mathcal{G} - \mathcal{E} , \quad y = \mathcal{B} - \mathcal{D} . \quad (13)$$

The first four equations appearing in the second column of (2) and two equations of (4) give the following equations for  $x$  and  $y$ :

$$y_\theta = 0 , \quad x_\varphi = -\cos \theta y , \quad y_\varphi = -\sin^2 \theta \partial_\theta \frac{x}{\sin \theta} . \quad (14)$$

These immediately imply  $y_{\varphi\varphi} + y = 0$  and hence

$$y = y_1 \cos \varphi + y_2 \sin \varphi , \quad x = \cos \theta y_\varphi + y_3 \sin \theta , \quad (15)$$

where three  $y_i$  functions depend on both  $t$  and  $r$ , and the expression of  $x$  is obtained by integration from (14).

### 1. The general forms of $\gamma, \epsilon$ and $\mu$

In terms of  $x$  and  $y$ , the last two equations of (2) are simply  $\mu_\theta = x$ ,  $\mu_\varphi = -y \sin \theta$ , and therefore they specify the general form of  $\mu$  as

$$\mu = y_\varphi \sin \theta - y_3 \cos \theta + c_0 T H_2 , \quad (16)$$

where  $c_0$  is an integration constant. In specifying the last term of (16), we make use of the  $\gamma_\theta$  and  $\epsilon_\theta$ -equations of (3), which transform to

$$\gamma_\theta = T^2 \frac{H_2}{H_0} \partial_t \frac{x_\theta}{T} , \quad \epsilon_\theta = \frac{H_2^2}{H_1} \partial_r \frac{x_\theta}{H_2} . \quad (17)$$

Integration of these equations with respect to  $\theta$  yield

$$\gamma = T^2 \frac{H_2}{H_0} \partial_t \frac{x}{T} + g(r, \varphi) , \quad \epsilon = \frac{H_2^2}{H_1} \partial_r \frac{x}{H_2} + \varepsilon(t, r, \varphi) . \quad (18)$$

By substituting these expression into  $x = \mathcal{G} - \mathcal{E}$ , we obtain

$$\varepsilon = \frac{\dot{T}}{H_0} \frac{g}{P} , \quad (19)$$

and a set of three equations for  $y_i$ , which in terms of  $Y_i = y_i/TH_2$  can be written as

$$Y_i = T\dot{T} \left( \frac{H_2}{H_0} \right)^2 Y_{it} - \frac{H_2^2 P}{H_1} Y_{ir} , \quad i = 1, 2, 3 . \quad (20)$$

In writing equations (20), we have equated the coefficient functions of different trigonometric functions forming a basis, and relation (19) results from the fact that  $x$  does not contain a term independent from  $\theta$ .

We now concentrate on specifying  $g$  and  $y_i$  functions. For this purpose we consider the coupled  $\epsilon_r$ -equation of (2) and  $\gamma_r$ -equation of (3) which, in terms of  $P$  and  $\ell$ , give

$$\partial_r \left( \frac{g}{P} \right) = H_1 g , \quad g_r = - \frac{\ell}{H_0^2} \frac{H_1}{P} g , \quad (21)$$

in addition to two sets of equations for  $y_i$  :

$$\partial_r \left( \frac{H_2^2}{H_1} Y_{ir} \right) = T\dot{T} \frac{H_1 H_2^2}{H_0^2} Y_{it} , \quad (22)$$

$$\partial_r (H_2^2 Y_{it}) = -H_2^2 Y_{itr} . \quad (23)$$

By dividing the first equation of (21) with  $P$ , it can be easily integrated to yield

$$g = H_2 P u_\varphi , \quad (24)$$

where for convenience the last factor of  $g$  has been written as the derivative of  $u = u(\varphi)$ . From the second equation of (21) we then obtain just one of the conditions of (11), when  $u_\varphi \neq 0$ .

After a slight rearrangement, the equation (23) can be integrated to find

$$Y_i = \frac{q_i(t)}{H_2} + \frac{z_i(r)}{H_2} \quad (25)$$

The substitution of this relation into (20) and (22) yield

$$T\dot{T}\dot{q}_i - [\ell + (1 - \ell_1) \frac{H_0^2}{H_2^2}] q_i = [\ell + (1 - \ell_1) \frac{H_0^2}{H_1}] z_i + H_0^2 \frac{P}{H_1} z'_i , \quad (26)$$

$$T\dot{T}\dot{q}_i - \ell q_i = \ell z_i - \frac{H_0^2 H_1'}{H_1^3} z'_i + \frac{H_0^2}{H_1^2} z''_i , \quad (27)$$

for  $i = 1, 2, 3$ . As we are about to see at the beginning of next section, the constant  $\ell_1$  must be 1. Anticipating this result here, we see that the above equations are separable, and therefore each side of them must be a constant such that

$$\begin{aligned} T\dot{T}\dot{q}_i - \ell q_i &= m_i , \\ \ell z_i - \frac{H_0^2 H_1'}{H_1^3} z_i' + \frac{H_0^2}{H_1^2} z_i'' &= m_i , \\ \ell z_i + \frac{H_0^2}{H_1} P z_i' &= m_i . \end{aligned} \tag{28}$$

It is not hard to see that the last two sets of (28) can be integrated to obtain

$$z_i = c_i H_2 P - \tilde{c}_i , \quad i = 1, 2, 3 \tag{29}$$

with  $m_i = -\ell \tilde{c}_i$ . The equations in the first line of (28) need not be integrated at this point, since they will be directly solved during the investigation of next subsection.

## 2. The general forms of $\alpha, \beta$ and $\delta$

Relations found by (25) specify  $y_i$  for  $i = 1, 2, 3$  as follows :

$$y_i = T q_i(t) + T z_i(r) . \tag{30}$$

By noting that  $(y_i/T)_{tr} = 0$ , the substitution of the expression of  $\gamma$  given by (18) into the second equation of (3) yield  $\alpha_\varphi = \ell_1 \sin \theta u_\varphi$  in view of (24). This can easily be integrated to write  $\alpha = \ell_1 \sin \theta u + f(\theta)$ , and then by using this in equations (8) and (9) we obtain

$$\begin{aligned} \ell_1(\ell_1 - 1) \sin \theta u + f_{\theta\theta} + \ell_1 f &= 0 , \\ \ell_1[u_{\varphi\varphi} + (\cos^2 \theta + \ell_1 \sin^2 \theta)u] + \ell_1 \sin \theta f + \cos \theta f_\theta &= 0 . \end{aligned}$$

Since  $u$  is a functions of  $\varphi$ , these relations imply that  $\ell_1$  must be either 0 or 1. In fact, the former value is not compatible with the very definition of spherical symmetry, since it would lead to metric coefficient functions depending on some finite powers of angular coordinates. Indeed, equations (8) and (9) show that when  $\ell_1 = 0$ ,  $\alpha$  can be assumed to be a nonzero constant. But the equations in the second column of (2) then show that there is no way to get rid of the above mentioned angular dependence. Therefore from now on, we take  $\ell_1 = 1$  by which the above two equations are reduced to the following forms :

$$f_{\theta\theta} + f = 0 , \quad u_{\varphi\varphi} + u + \sin \theta f + \cos \theta f_\theta = 0 . \tag{31}$$



Hence, in terms of integration constants  $c_4, c_5, c_6, \tilde{c}_4$  and

$$v = v(\varphi) = c_5 \cos \varphi + c_6 \sin \varphi , \quad (32)$$

we have  $f = c_4 \cos \theta + \tilde{c}_4 \sin \theta$  and  $u = v - \tilde{c}_4$  which implies  $u_\varphi = v_\varphi$ . These completely specify  $\alpha$  as

$$\alpha = c_4 \cos \theta + \sin \theta v . \quad (33)$$

Using (33) in the second equation of (2) and the first equation of (3) leads us to

$$\beta = H_2 P \alpha_\theta + H_2 \bar{f}(\varphi) , \quad \delta = \dot{T} \frac{H_2}{H_0} (\alpha_\theta + \frac{\bar{f}}{P}) - \frac{y}{H_2 P} , \quad (34)$$

where  $\delta$  is obtained from the definition  $y = \mathcal{B} - \mathcal{D}$ . Substitutions of the solutions (33) and (34) into the  $\gamma_\varphi$ -equation of (2) yield, in terms of constants  $c_7$  and  $c_8$

$$\bar{f} = \frac{v_1}{H_0} , \quad v_1 = v_1(\varphi) = c_7 \cos \varphi + c_8 \sin \varphi , \quad (35)$$

and  $\dot{q}_1 = c_7/T^2$ ,  $\dot{q}_2 = c_8/T^2$  since  $\gamma$  is independent of time.

### 3. KY 2-forms for time-dependent de Sitter space-time: $\ell \neq 0$ solutions

From the first set of equations of (28),  $q_1$  and  $q_2$  can be completely specified as

$$q_1 = c_7 \frac{\dot{T}}{\ell T} + \tilde{c}_1 , \quad q_2 = c_8 \frac{\dot{T}}{\ell T} + \tilde{c}_2 , \quad (36)$$

by recalling the relations  $m_1 = -\ell \tilde{c}_1$  and  $m_2 = -\ell \tilde{c}_2$ . In view of (29), (30) and (36),  $y$  is also completely specified as

$$y = \frac{\dot{T}}{\ell} v_1 + T H_2 P v_2 , \quad (37)$$

where we have defined

$$v_2 = v_2(\varphi) = c_1 \cos \varphi + c_2 \sin \varphi . \quad (38)$$

In view of (35) and (37), while  $\beta$  and  $\delta$  have been completely determined, there remains to determine  $q_3(t)$  of the  $y_3$ -function

$$y_3 = T q_3(t) + T(c_3 H_2 P - \tilde{c}_3) \quad (39)$$

for complete specification of  $\gamma, \epsilon$  and  $\mu$ . The resulting  $\beta, \delta$  solutions identically satisfy the  $\beta_r$ -equation of (3) and  $\delta_r$ -equation of (2), and therefore they give nothing new. On the other hand, if (37) and (38) are used in the  $\gamma$ -expression of (18), we see that like  $q_1$  and  $q_2$ ,  $q_3$  must be equal to  $c_9(\dot{T}/\ell T) + \tilde{c}_3$  since  $\gamma$  does not depend on  $t$ . The resulting  $\gamma$  and  $\epsilon$  also satisfy the  $\epsilon_r$ -equation of (2), as well as the definition  $x = \mathcal{G} - \mathcal{E}$ . But, as they do not take part in any metric coefficient functions, the constants  $\tilde{c}_1, \tilde{c}_2$  and  $\tilde{c}_3$  become redundant. We are now ready to present all the coefficient functions together :

$$\begin{aligned}
\alpha &= c_4 \cos \theta + \sin \theta v, \quad \beta = H_2 P \alpha_\theta + \frac{H_2}{H_0} v_1, \\
\gamma &= \frac{H_2}{H_0} (c_9 \sin \theta + \cos \theta v_{1\varphi}) + H_2 P v_\varphi, \\
\delta &= \dot{T} \frac{H_2}{H_0} \alpha_\theta - \frac{1}{\ell} \dot{T} H_2 P v_1 - T v_2, \\
\epsilon &= -\sin \theta (c_9 \frac{1}{\ell} \dot{T} H_2 P + c_3 T) - \cos \theta (\frac{1}{\ell} \dot{T} H_2 P v_{1\varphi} + T v_{2\varphi}) + \dot{T} \frac{H_2}{H_0} v_\varphi, \\
\mu &= \sin \theta (\frac{1}{\ell} \dot{T} v_{1\varphi} + T H_2 P v_{2\varphi}) - \cos \theta (\frac{c_9}{\ell} \dot{T} + c_3 T H_2 P) + c_0 T H_2,
\end{aligned} \tag{40}$$

which determine ten linearly independent KY 2-forms, one for each  $c_i$ ,  $i = 0, 1, \dots, 9$ :

$$\begin{aligned}
\omega_0 &= T H_2 e^{23}, \\
\omega_1 &= -T \cos \varphi e^{12} + T \sin \varphi (\cos \theta e^{13} + H_2 P \sin \theta e^{23}), \\
\omega_2 &= -T \sin \varphi e^{12} - T \cos \varphi (\cos \theta e^{13} - H_2 P \sin \theta e^{23}), \\
\omega_3 &= -T (\sin \theta e^{13} + H_2 P \cos \theta e^{23}), \\
\omega_4 &= \cos \theta e^{01} - H_2 \sin \theta \Omega_1 \wedge e^2, \\
\omega_5 &= \cos \varphi (\sin \theta e^{01} + H_2 \cos \theta \Omega_1 \wedge e^2) - H_2 \sin \varphi \Omega_1 \wedge e^3, \\
\omega_6 &= \sin \varphi (\sin \theta e^{01} + H_2 \cos \theta \Omega_1 \wedge e^2) + H_2 \cos \varphi \Omega_1 \wedge e^3, \\
\omega_7 &= H_2 \cos \varphi \Omega_2 \wedge e^2 - H_2 \sin \varphi (\cos \theta \Omega_2 \wedge e^3 + \frac{\dot{T}}{\ell} \sin \theta e^{23}), \\
\omega_8 &= H_2 \sin \varphi \Omega_2 \wedge e^2 + H_2 \cos \varphi (\cos \theta \Omega_2 \wedge e^3 + \frac{\dot{T}}{\ell} \sin \theta e^{23}), \\
\omega_9 &= H_2 \sin \theta \Omega_2 \wedge e^3 - \frac{\dot{T}}{\ell} \cos \theta e^{23}.
\end{aligned} \tag{41}$$

where the 1-forms  $\Omega_1$  and  $\Omega_2$  are defined, for the sake of simplicity, as

$$\Omega_1 = P e^0 + \frac{\dot{T}}{H_0} e^1, \quad \Omega_2 = \frac{1}{H_0} e^0 - \frac{\dot{T}}{\ell} P e^1. \tag{42}$$

#### 4. KY 2-forms for the usual form of de Sitter space-time: $\ell = 0$ solutions

When  $\ell$  is zero, the integrability conditions (11) and condition (10) for  $\ell_1 = 1$  imply that

$$H_2 P = \varepsilon, \quad P' = -\frac{H_1}{H_2^2}, \quad \dot{T} = \lambda T, \quad (43)$$

where  $\varepsilon$  and  $\lambda$  are some nonzero constants. One can easily verify that the first two equations of (43) yield  $\varepsilon^2 = 1$ , therefore  $\varepsilon = \pm 1$ , and that the first relation implies the second one.

The nine equations of (28) are then as follows for  $i = 1, 2, 3$ :

$$\dot{q}_i = \frac{m_i}{\lambda T^2}, \quad z'_i = \frac{m_i}{H_0^2} \frac{H_1}{P}, \quad z''_i - \frac{H'_1}{H_1} z'_i = \frac{m_i}{H_0^2} H_1^2. \quad (44)$$

The first two sets can be easily solved, such that  $m_i = 2c_i H_0^2$  and

$$q_i = -c_i \frac{H_0^2}{\lambda^2 T^2} + a_i, \quad z_i = c_i H_2^2 + b_i, \quad (45)$$

where  $a_i$  and  $b_i$  are new integration constants. These  $z_i$  solutions identically satisfy the last relations of (44) without any extra condition.

By defining  $d_i = a_i + b_i$  and

$$v_3 = v_3(\varphi) = d_1 \cos \varphi + d_2 \sin \varphi, \quad (46)$$

$y_i$  and hence  $x, y$  functions can be explicitly written from (15) and (30) as

$$y_i = T(c_i S_- + d_i), \quad y = T(S_- v_2 + v_3), \quad (47)$$

$$x = T(S_- v_{2\varphi} + v_{3\varphi}) \cos \theta + T(c_3 S_- + d_3) \sin \theta, \quad (48)$$

where  $v_2$  is given by the relation (38) and

$$S_{\pm} = H_2^2 \pm \frac{H_0^2}{\lambda^2 T^2}. \quad (49)$$

The substitution of  $\alpha$  and  $\beta$  given by (33) and (34) into the  $\gamma_{\varphi}$ -equation of (2) yields

$$\bar{f}(\varphi) = \frac{T^2}{H_0} \partial_t \frac{y}{T} = 2 \frac{H_0}{\lambda} v_2. \quad (50)$$

Using (38), (47) and (48) in (18) and (34) give the following:

$$\begin{aligned} \alpha &= c_4 \cos \theta + \sin \theta v, \quad \beta = H_2 P \alpha_{\theta} + 2 \frac{H_0}{\lambda} H_2 v_2, \\ \gamma &= 2 \frac{H_0}{\lambda} H_2 (c_3 \sin \theta + \cos \theta v_{2\varphi}) + H_2 P v_{\varphi}, \\ \delta &= \frac{\lambda}{H_0} T H_2 \alpha_{\theta} + T \frac{1}{H_2 P} (S_+ v_2 - v_3), \\ \epsilon &= \varepsilon T S_+ (c_3 \sin \theta + \cos \theta v_{2\varphi}) - \varepsilon T (d_3 \sin \theta + \cos \theta v_{3\varphi}) + \frac{\lambda}{H_0} T H_2 v_{\varphi}, \\ \mu &= T (S_- v_{2\varphi} + v_{3\varphi}) \sin \theta - T (d_3 + c_3 S_-) \cos \theta + c_0 T H_2, \end{aligned} \quad (51)$$

which determine ten linearly independent KY 2-forms, one for each  $c_i$ ,  $i = 0, 1, \dots, 6$  and  $d_j$ ,  $j = 1, 2, 3$ . The corresponding KY 2-forms can be written as

$$\begin{aligned}
\omega_0 &= TH_2 e^{23}, \\
\omega_1 &= 2\frac{H_0}{\lambda} H_2 e^0 \wedge \Phi_\varphi + \varepsilon TS_+ e^1 \wedge \Phi - TS_- \sin \theta \sin \varphi e^{23}, \\
\omega_2 &= 2\frac{H_0}{\lambda} H_2 e^0 \wedge \Phi - \varepsilon TS_+ e^1 \wedge \Phi_\varphi + TS_- \sin \theta \cos \varphi e^{23}, \\
\omega_3 &= 2\frac{H_0}{\lambda} H_2 \sin \theta e^{03} + \varepsilon TS_+ \sin \theta e^{13} - TS_- \cos \theta e^{23}, \\
\omega_4 &= \cos \theta e^{01} - \sin \theta \Psi \wedge e^2, \\
\omega_5 &= \cos \varphi (\sin \theta e^{01} + \cos \theta \Psi \wedge e^2) - \sin \varphi \Psi \wedge e^3, \\
\omega_6 &= \sin \varphi (\sin \theta e^{01} + \cos \theta \Psi \wedge e^2) + \cos \varphi \Psi \wedge e^3, \\
\omega_7 &= -\varepsilon T \cos \varphi e^{12} + T \sin \varphi (\varepsilon \cos \theta e^{13} - \sin \theta e^{23}), \\
\omega_8 &= -\varepsilon T \sin \varphi e^{12} - T \cos \varphi (\varepsilon \cos \theta e^{13} - \sin \theta e^{23}), \\
\omega_9 &= -\varepsilon T \sin \theta e^{13} - T \cos \theta e^{23},
\end{aligned} \tag{52}$$

where the 1-forms  $\Phi$  and  $\Psi$  are defined, for the sake of simplicity, as

$$\Phi = \cos \varphi e^2 - \cos \theta \sin \varphi e^3, \quad \Psi = \varepsilon e^0 + \frac{\lambda}{H_0} TH_2 e^1, \tag{53}$$

and  $\Phi_\varphi = -(\sin \varphi e^2 + \cos \theta \cos \varphi e^3)$ .

#### IV. KY 2-FORMS FOR $\dot{T} = 0$ , $H'_0 \neq 0$ , $\alpha \neq 0$

In this case, condition (7) requires

$$H'_2 = kH_0H_1, \tag{54}$$

such that  $k$  is a nonzero constant and, in addition to  $\alpha_\tau = 0 = \alpha_r$ , we immediately have  $\delta_r = 0 = \delta_\theta$  and  $\epsilon_r = 0$  from equations (2). Moreover, the  $\epsilon_\varphi$ -equation of (2) becomes independent of the metric characterizing functions, and leads to  $\delta_{\varphi\varphi} + \delta = 0$  when substituted into the  $\varphi$ -derivative of the second equation of (4). Therefore,  $\delta$  is a harmonic function of  $\varphi$  and when this fact is used in the integration of  $\epsilon_\varphi = -\cos \theta \delta$ , we obtain

$$\begin{aligned}
\delta &= D_1(\tau) \cos \varphi + D_2(\tau) \sin \varphi, \\
\epsilon &= \cos \theta \delta_\varphi + E(\tau) \sin \theta, \\
\mu &= -H_2 P \sin \theta \delta_\varphi + E(\tau) H_2 P \cos \theta + U(\tau, r),
\end{aligned} \tag{55}$$

where we have also made use of the  $\mu_\theta$  and  $\mu_\varphi$ -equations of (2). The  $\epsilon_\theta$ -equation then yields  $P' = -H_1/H_2^2$ , which means that  $\ell_1 = 1$  and  $U = H_2 u(\tau)$ . Therefore,  $\alpha$  is again given by (35) and  $\beta$  can be written, from the second equation of (2) and the first equation of (3), as  $\beta = H_2 P \alpha_\theta + H_2 B(\tau, \varphi)$ . It remains to determine only the  $\tau$  dependence of  $\delta, \epsilon$  and  $\mu$ .

The  $\beta_\tau$ -equation of (2) and the third equation of (3) give

$$B_\tau = m\delta, \quad m_1 B = -\delta_\tau, \quad (56)$$

provided that  $m$  and  $m_1$  defined by

$$m = \frac{H'_0}{H_1 H_2}, \quad m_1 = \left(\frac{H_2}{H_0}\right)' \frac{H_0^2}{H_1} = k H_0^2 - m H_2^2, \quad (57)$$

are new constants, such that  $m$  is supposed to be nonzero in this section. Thus,  $\delta$  must satisfy  $\delta_{\tau\tau} + m m_1 \delta = 0$ , and therefore its coefficient functions can be obtained, depending on the value of  $m m_1$ , from

$$D_{i\tau\tau} + m m_1 D_i = 0, \quad i = 1, 2. \quad (58)$$

From here on, the discussion proceeds in two ways: (A)  $m_1 \neq 0$  and (B)  $m_1 = 0$ .

#### A. KY 2-Forms of the static de Sitter space-time: $m_1 \neq 0$ Solutions

In this case, the  $\gamma_\varphi$ -equation of (2), the second equation of (3) and the first equation of (4) specify  $\gamma$  as

$$\gamma = H_2 P v_\varphi - \frac{H_2}{m_1} \cos \theta \delta_{\tau\varphi} + H_2 \sin \theta g(\tau). \quad (59)$$

Only three equations remain unused so far; the  $\gamma_\tau, \gamma_\theta$ -equations and the equation  $\partial_r(\gamma/H_0) = -H_1 \epsilon_\tau/H_0^2$  of (3). One can easily check that the first two of these equations yield

$$g_\tau = mE, \quad g = -kE_\tau, \quad k m_1 = 1, \quad u_\tau = 0, \quad (60)$$

and the last equation gives nothing new. In accordance with the previous solutions, we take the constant  $u$  as  $u = c_0 T$ . The first two equations of (60) also give  $E_{\tau\tau} + m m_1 E = 0$ . We can now write the complete solutions of the case:

$$\alpha = c_4 \cos \theta + \sin \theta v, \quad \beta = H_2 P \alpha_\theta - \frac{H_2}{m_1} \delta_\tau,$$

$$\begin{aligned}
\gamma &= H_2 P v_\varphi - \frac{H_2}{m_1} (\cos \theta \delta_{\tau\varphi} + \sin \theta E_\tau) , \\
\delta &= D_1 \cos \varphi + D_2 \sin \varphi , \\
\epsilon &= \cos \theta \delta_\varphi + E \sin \theta , \\
\mu &= -H_2 P \sin \theta \delta_\varphi + E H_2 P \cos \theta + c_0 T H_2 ,
\end{aligned} \tag{61}$$

where for, say  $mm_1 = -w_0^2 < 0$ , we have

$$\begin{aligned}
D_1 &= a_1 \cosh w_0 \tau + a_2 \sinh w_0 \tau , \\
D_2 &= a_3 \cosh w_0 \tau + a_7 \sinh w_0 \tau , \\
E &= a_8 \cosh w_0 \tau + a_9 \sinh w_0 \tau .
\end{aligned} \tag{62}$$

When  $mm_1 = w_0^2 > 0$ , it is enough to replace the hypergeometric functions of (62) by the corresponding trigonometric functions and the minus sign by a plus in the expression of  $H_0^2$ . The above solutions define ten linearly independent KY 2-forms

$$\begin{aligned}
\omega_0 &= T H_2 e^{23} , \\
\omega_1 &= -\frac{w_0}{m_1} H_2 \sinh w_0 \tau e^0 \wedge \Phi + \cosh w_0 \tau (e^1 \wedge \Phi + H_2 P \sin \theta \sin \varphi e^{23}) , \\
\omega_2 &= -\frac{w_0}{m_1} H_2 \cosh w_0 \tau e^0 \wedge \Phi + \sinh w_0 \tau (e^1 \wedge \Phi + H_2 P \sin \theta \sin \varphi e^{23}) , \\
\omega_3 &= \frac{w_0}{m_1} H_2 \sinh w_0 \tau e^0 \wedge \Phi_\varphi - \cosh w_0 \tau (e^1 \wedge \Phi_\varphi + H_2 P \sin \theta \cos \varphi e^{23}) , \\
\omega_4 &= \cos \theta e^{01} - H_2 P \sin \theta e^{02} , \\
\omega_5 &= \sin \theta \cos \varphi e^{01} + H_2 P (\cos \theta \cos \varphi e^{02} - \sin \varphi e^{03}) , \\
\omega_6 &= \sin \theta \sin \varphi e^{01} + H_2 P (\cos \theta \sin \varphi e^{02} + \cos \varphi e^{03}) , \\
\omega_7 &= \frac{w_0}{m_1} H_2 \cosh w_0 \tau e^0 \wedge \Phi_\varphi - \sinh w_0 \tau (e^1 \wedge \Phi_\varphi + H_2 P \sin \theta \cos \varphi e^{23}) , \\
\omega_8 &= -\frac{w_0}{m_1} H_2 \sinh w_0 \tau \sin \theta e^{03} + \cosh w_0 \tau (\sin \theta e^1 + H_2 P \cos \theta e^2) \wedge e^3 , \\
\omega_9 &= -\frac{w_0}{m_1} H_2 \cosh w_0 \tau \sin \theta e^{03} + \sinh w_0 \tau (\sin \theta e^1 + H_2 P \cos \theta e^2) \wedge e^3 ,
\end{aligned} \tag{63}$$

corresponding, respectively, to  $c_i$ ,  $i = 0, 4, 5, 6$  and  $a_j$ ,  $j = 1, 2, 3, 7, 8, 9$ . Here,  $\Phi$  is given by (53) and the metric coefficient functions are as follows:

$$H_0^2 = m_1^2 - w_0^2 H_2^2 , \quad H_1 = m_1 \frac{H_2'}{H_0} . \tag{64}$$

## B. $m_1 = 0$ Solutions

When  $m_1$  is zero, we have  $H_2 = m_0 H_0$  where  $m_0$  is a nonzero constant and, by virtue of (54) and (57),  $k = mm_0^2$ . Therefore  $k$  and  $m$  have the same sign, and (54) and (57) amount to the same relation. In that case, the equations of (56) are of the forms  $\delta_\tau = 0$  and  $B_\tau = m\delta$ , which imply that  $B = m\tau\delta + C_\varphi(\varphi)$  and

$$\beta = H_2(P\alpha_\theta + m\tau\delta + C_\varphi) , \quad \delta = b_1 \cos \varphi + b_2 \sin \varphi . \quad (65)$$

where  $b_i$ 's are constants. The  $\gamma_\varphi$ -equation of (2) and the second equation of (3) provide us with

$$\gamma = H_2[Pv_\varphi + \cos \theta(m\tau\delta_\varphi - C(\varphi)) + G(\tau, \theta)] , \quad (66)$$

such that  $\ell_1 = 1$ . Thus,  $\alpha$  is still given by (33) and  $\epsilon, \mu$  are as in equations (55). For the above solutions, we have  $\partial_r(\gamma/H_0) = 0$  and the fifth equation of (3) implies that  $\epsilon_\tau = 0$ , that is,  $E$  is independent of time. On the other hand, the  $\gamma_\tau$ -equation of (2) and the first equation of (4) yield

$$G_\tau = mE \sin \theta , \quad C_{\varphi\varphi} = -\sin^2 \theta \partial_\theta \frac{G}{\sin \theta} .$$

We are finally left with the  $\gamma_\theta$ -equation, which gives

$$G_\theta = \sin \theta(m\tau\delta_\varphi - C) - \frac{H_2}{H_0} u_\tau .$$

As  $G$  is independent of  $\varphi$ , the only possible solutions of the last three equations are  $G_\theta = -c \sin \theta$  and  $\delta_\varphi = 0 = u_\tau$  such that  $C = c = \text{constant}$ . Since  $G_{\tau\theta} = G_{\theta\tau}$  implies  $E = 0$ , we can write  $\epsilon = 0 = \delta$  and  $u = c_0 T$ . Although  $G = c \cos \theta$  is a solution,  $c$  does not take part in any component functions. The results can therefore be written as follows:

$$\begin{aligned} \alpha &= c_4 \cos \theta + \sin \theta v , & \beta &= H_2 P \alpha_\theta , \\ \gamma &= H_2 v_\varphi , & \delta &= 0 = \epsilon , & \mu &= c_0 T H_2 , \end{aligned}$$

which define four linearly independent KY 2-forms;  $\omega_0$  and

$$\begin{aligned} \omega_1 &= \cos \theta e^{01} - mm_0 H_2 \sin \theta e^{02} , \\ \omega_2 &= -\cos \varphi \omega_{1\theta} - H_2 \sin \varphi e^{03} , \\ \omega_3 &= -\sin \varphi \omega_{1\theta} + H_2 \cos \varphi e^{03} . \end{aligned}$$

(The corresponding 1-forms can be found from Section IV-A or V with  $k_1 = 0$  of the first paper).

## V. KY 2-FORMS OF THE FLAT SPACE-TIME

In this section we take both  $\dot{T}$  and  $H'_0$  to be zero, which imply  $\beta_\tau = 0 = \gamma_\tau$ ,  $\delta_r = 0 = \delta_\theta$  and  $\epsilon_r = 0$ . Thus, as in the previous section, we have

$$\begin{aligned}\beta &= H_2 P \alpha_\theta + H_2 B(\varphi) , \\ \delta &= D_1(\tau) \cos \varphi + D_2(\tau) \sin \varphi , \\ \epsilon &= \cos \theta \delta_\varphi + E(\tau) \sin \theta , \quad \mu = H_2(P\epsilon_\theta + u(\tau)) ,\end{aligned}\tag{67}$$

provided that  $P' = -H_1/H_2^2$  and hence  $\alpha$  is given by (33). On the other hand, the  $\beta_r$ -equation of (3) gives

$$(H_2 P)' = 0 , \quad B = -\frac{H_1}{H_0 H_2'} \delta_\tau .\tag{68}$$

The first relation is equivalent to the first integrability condition of (7), and when it is combined with  $P' = -H_1/H_2^2$ , we obtain  $P = \varepsilon/H_2$ , that is  $\varepsilon H_2' = H_1$  with  $\varepsilon = \pm 1$ . Five equations remain untouched so far; the  $\gamma_\varphi$ -equation of (2), three equations of (3) involving  $\gamma$ , and the first equation of (4).

We first consider the  $\gamma_\theta$ -equation of (3) by inserting the  $\mu$ -solution of (67) into it. The resulting equation implies that  $u$  must be a constant, which we take to be  $u = c_0 T$ , and we are left with

$$\gamma_\theta = \frac{H_2^2 P}{H_0} (\sin \theta \delta_{\tau\varphi} - E_\tau \cos \theta) .\tag{69}$$

As  $\gamma$  is independent of  $\tau$ ,  $E_\tau$  and  $D_{j\tau}$  must be constants, and therefore  $E = a_0 \tau + b_0$ ,  $D_1 = b_1 \tau + b_3$  and  $D_2 = b_2 \tau + b_4$ , where  $a_i$  and  $b_j$  are constants. Thus, in terms of

$$z_1(\varphi) = b_1 \cos \varphi + b_2 \sin \varphi , \quad z_2(\varphi) = b_3 \cos \varphi + b_4 \sin \varphi ,\tag{70}$$

we can write  $\delta = \tau z_1 + z_2$ , which implies that  $B = -z_1/H_0 H_2 P$  by virtue of the second relation of (68). Therefore, (69) can be integrated to obtain the explicit expression of  $\gamma$ , presented together with the other solutions below:

$$\begin{aligned}\alpha &= c_4 \cos \theta + \sin \theta v , \quad \beta = H_2 P \alpha_\theta - \frac{1}{H_0 P} z_1 , \\ \gamma &= -\frac{H_2^2 P}{H_0} (\cos \theta z_{1\varphi} + a_0 \sin \theta) + H_2 P v_\varphi , \\ \delta &= \tau z_1 + z_2 , \\ \epsilon &= \tau (\cos \theta z_{1\varphi} + a_0 \sin \theta) + \cos \theta z_{2\varphi} + b_0 \sin \theta , \\ \mu &= H_2 (P \epsilon_\theta + c_0 T) .\end{aligned}\tag{71}$$



The last term of  $\gamma$  which arises from the  $\theta$ -integration mentioned above is specified by the  $\beta_\varphi$ -equation of (4). It is straightforward to verify that the remaining three equations of  $\gamma$  are identically satisfied. The corresponding KY 2-forms are, in addition to  $\omega_0$ , as follows:

$$\begin{aligned}
\omega_1 &= -\varepsilon \frac{H_2}{H_0} (\cos \varphi e^{02} - \cos \theta \sin \varphi e^{03}) + \tau (\cos \varphi e^{12} - \sin \varphi A \wedge e^3) , \\
\omega_2 &= -\varepsilon \frac{H_2}{H_0} (\sin \varphi e^{02} + \cos \theta \cos \varphi e^{03}) + \tau (\sin \varphi e^{12} + \cos \varphi A \wedge e^3) , \\
\omega_3 &= \cos \varphi e^{12} - \sin \varphi A \wedge e^3 , \quad \omega_4 = e^0 \wedge A , \\
\omega_5 &= -\cos \varphi e^0 \wedge A_\theta - \varepsilon \sin \varphi e^{03} , \\
\omega_6 &= -\sin \varphi e^0 \wedge A_\theta + \varepsilon \cos \varphi e^{03} , \quad \omega_7 = \sin \varphi e^{12} + \cos \varphi A \wedge e^3 , \\
\omega_8 &= -\varepsilon \frac{H_2}{H_0} \sin \theta e^{03} - \tau A_\theta \wedge e^3 , \quad \omega_9 = -A_\theta \wedge e^3 ,
\end{aligned} \tag{72}$$

where  $A = \cos \theta e^1 - \varepsilon \sin \theta e^2$ .

## VI. SOLUTIONS FOR $\alpha = 0$

In this case the second and fourth equations of (2) give  $\beta_\theta = 0 = \delta_\theta$  and

$$\beta = H_2 B(t, \varphi) , \quad \gamma = H_2 G(t, \theta, \varphi) , \tag{73}$$

are implied by the first two equations of (3). Fourteen equations remain to be solved. But when  $\alpha$  is set to zero, the  $\gamma_\varphi$  and  $\epsilon_\varphi$ -equations of (2) are freed from metric coefficient functions, such that  $\gamma_\varphi = -\cos \theta \beta$  and  $\epsilon_\varphi = -\cos \theta \delta$ . When these are combined with two equations of (4), they considerably ease the investigation by providing general forms of the solutions. Indeed, differentiating both equations of (4) with respect to  $\varphi$  gives  $\beta_{\varphi\varphi} + \beta = 0 = \delta_{\varphi\varphi} + \delta$ . Therefore, the general forms of  $\beta, \gamma, \delta, \epsilon$  and  $\mu$  can be written, in view of (73), as follows:

$$\begin{aligned}
\beta &= H_2 (B_1(t) \cos \varphi + B_2(t) \sin \varphi) , \\
\gamma &= \cos \theta \beta_\varphi + G(t) H_2 \sin \theta , \\
\delta &= D_1(t, r) \cos \varphi + D_2(t, r) \sin \varphi , \\
\epsilon &= \cos \theta \delta_\varphi + E(t, r) \sin \theta , \\
\mu &= (\dot{T} \frac{H_2}{H_0} \beta_\varphi - \frac{H'_2}{H_1} \delta_\varphi) \sin \theta + U(t, r, \theta) ,
\end{aligned} \tag{74}$$

where functions  $B_i, D_i, G, E$  and  $U$  are to be determined from the remaining nine equations: the five equations in the first column of (2) and the last four equations of (3). The second and fourth relations are obtained by integrations and then by using the results in equations (4). The last relation of (74) is obtained by first using  $\beta$  and  $\delta$  obtained in the last relation of (2) and then by integrating the resulting equation with respect to  $\varphi$ .

When the solutions (74) are substituted into the five equations appearing in the first column of equations (2), we obtain

$$\begin{aligned}\dot{B}_1 &= \frac{M}{T}D_1, & \dot{B}_2 &= \frac{M}{T}D_2, \\ D_{1r} &= \dot{T}LB_1, & D_{2r} &= \dot{T}LB_2, \\ \dot{G} &= \frac{M}{T}E, & E_r &= \dot{T}LG, \\ U_\theta &= (\dot{T}\frac{H_2^2}{H_0}G - \frac{H_2'}{H_1}E)\sin\theta,\end{aligned}\tag{75}$$

where we have defined the functions

$$M = \frac{H_0'}{H_1H_2}, \quad L = \frac{H_1H_2}{H_0}.\tag{76}$$

### A. A General Case

To be as general as possible, we shall first seek solutions for which both  $\dot{T}$  and  $H_0'$  can be different from zero. In such a case, provided that  $m$  defined by

$$m = \frac{M'}{M^2L}\tag{77}$$

is a constant, from the first four equations of (75) one can easily obtain

$$B_i = b_i K, \quad D_i = b_i \frac{\dot{K}}{K^m M}; \quad i = 1, 2,\tag{78}$$

where  $b_1$  and  $b_2$  are integration constants,  $K = T^{-1/m}$  and  $m$  is supposed to be different from zero. In a similar way, the fifth and sixth equations of (75) yield

$$G = b_3 K, \quad E = b_3 \frac{\dot{K}}{K^m M}, \quad U_\theta = -b_3 \frac{\dot{K}R}{K^m M} \sin\theta,\tag{79}$$

where  $b_3$  is a constant, and we have defined

$$R = \frac{H_2}{H_1} \left( m \frac{H_0'}{H_0} + \frac{H_2'}{H_2} \right).\tag{80}$$

In terms of  $g = b_1 \cos \varphi + b_2 \sin \varphi$ , the solutions can now be rewritten as

$$\begin{aligned}\beta &= KH_2 g, & \gamma &= KH_2(g_\varphi \cos \theta + b_3 \sin \theta), \\ \delta &= \frac{\dot{K}}{K^m M} g, & \epsilon &= \frac{\dot{K}}{K^m M}(g_\varphi \cos \theta + b_3 \sin \theta), \\ \mu &= -\frac{\dot{K}R}{K^m M}(g_\varphi \sin \theta - b_3 \cos \theta) + u(t, r),\end{aligned}\tag{81}$$

where the integration of  $U_\theta$  done with respect to  $\theta$ . There remain the last four equations of (3) that have not been used so far. Substitution of these solutions into the last four equations of (3) yield

$$\begin{aligned}\ddot{K} &= -m_1 K^{2m+1} = -m_2 K^{2m+1}, \\ \partial_t\left(\frac{u}{T}\right) &= 0 = \partial_r\left(\frac{u}{H_2}\right), \\ \frac{H_1}{H_2^2} &= -M\left(\frac{R}{H_2 M}\right)',\end{aligned}\tag{82}$$

provided that  $m_1$  and  $m_2$  defined by

$$m_1 = M \frac{H_0^2}{H_1} \left(\frac{H_2}{H_0}\right)', \quad m_2 = M \frac{H_0}{R},\tag{83}$$

are constant. In fact, the first equality of (82) implies that  $m_1 = m_2$ , and

$$\ddot{K} = -m_1 K^{2m+1}, \quad R \frac{H_0}{H_1} \left(\frac{H_2}{H_0}\right)' = 1.\tag{84}$$

The equations appearing in the second line of (82) give  $u = TH_2$ , and hence  $\mu$  has been completely specified as

$$\mu = -\frac{\dot{K}R}{K^m M}(g_\varphi \sin \theta - b_3 \cos \theta) + c_0 TH_2.\tag{85}$$

We have obtained five linearly independent KY 2-forms for a family of space-times characterized by two constants  $m$  and  $m_1$ .

Having determined the coefficient functions of  $\omega_2$ , we now turn to the conditions which restrict the functions determining the metric tensor. In addition to two conditions given by (84), we have two more conditions defining the constants  $m$  and  $m_1$ . The first, given by (77), can be integrated to yield

$$H_0' = k H_0^{-1/m} H_1 H_2\tag{86}$$

and hence,  $M = kH_0^m$ . From the first equation of (83) and the second of (84), we find

$$R = \frac{k}{m_1} H_0^{m+1} , \quad H_2' = kH_0^{m-1} H_2^2 + \frac{m_1}{k} H_1 H_0^{-m-1} .$$

Substitutions of these into (73) finally yields

$$m_1 H_2 \left( m + \frac{1}{H_1} \right) = H_0^2 - \left( \frac{m_1}{k} \right)^2 H_0^{-2m} . \quad (87)$$

Although several subclasses of space-times can be identified for particular values of  $m$  and  $m_1$ , this general consideration will not be pursued any further. It will suffice to exhibit the physically important final case.

## B. KY 2-forms of the Robertson-Walker space-time

The Robertson-Walker space-time is characterized by

$$H_0 = 1 , \quad H_2 = r , \quad H_1^2 = \frac{1}{1 + k_3 r^2} , \quad (88)$$

such that  $T$  is specified by special cosmological models. Two such specifications are  $T = t^{1/2}$  and  $T = t^{2/3}$  which correspond, respectively, to radiation-dominated and matter-dominated universes. In this final subsection, we shall present KY 2-forms of this space-time such that there is no constraint on  $T$ . It turns out that such a case is possible only if we take  $\alpha = 0 = \beta = \gamma$  and  $H_0' = 0$ . The solutions then are

$$\begin{aligned} \delta &= T(c_1 \cos \varphi + c_2 \sin \varphi) , \\ \epsilon &= \cos \theta \delta_\varphi + c_3 T \sin \theta , \\ \mu &= -H_2 P \sin \theta \delta_\varphi + T(c_3 H_2 P \cos \theta + c_0 H_2) , \end{aligned} \quad (89)$$

provided that  $P' = -H_1/H_2^2$ , which leads to  $H_1$  of (94). The above solutions provide us, in addition to  $\omega_0$ , with three linearly independent KY 2-forms:

$$\begin{aligned} \omega_1 &= T(\cos \varphi e^{12} - \cos \theta \sin \varphi e^{13}) + H_2 P \sin \theta \sin \varphi e^{23} , \\ \omega_2 &= T(\sin \varphi e^{12} + \cos \theta \cos \varphi e^{13}) - H_2 P \sin \theta \cos \varphi e^{23} , \\ \omega_3 &= T(\sin \theta e^{13} + H_2 P \cos \theta e^{23}) . \end{aligned} \quad (90)$$

## VII. KY 3-FORMS

For a 3-form

$$\omega_{(3)} = \alpha e^{012} + \beta e^{013} + \gamma e^{023} + \delta e^{123} ,$$

the KY-equation gives sixteen equations, five of which have the following simple forms:

$$\alpha_t = 0 = \beta_t , \quad \alpha_r = 0 = \beta_r , \quad \alpha_\theta = 0 .$$

These imply that  $\alpha$  depends only on  $\varphi$ , and  $\beta$  is a function of  $\theta$  and  $\varphi$ . Seven of the remaining equations have the following two-term forms:

$$\begin{aligned} \beta_\varphi &= -\cos \theta \alpha , \\ \gamma_t &= \frac{H'_0}{T H_1} \delta , \quad \gamma_\theta = -\frac{H'_2}{H_1} \beta , \quad \gamma_\varphi = \frac{H'_2}{H_1} \sin \theta \alpha , \\ \delta_r &= \dot{T} \frac{H_1}{H_0} \gamma , \quad \delta_\theta = -\dot{T} \frac{H_2}{H_0} \beta , \quad \delta_\varphi = \dot{T} \frac{H_2}{H_0} \sin \theta \alpha . \end{aligned} \tag{91}$$

The last four equations give three independent equations, which can be written as:

$$\alpha_\varphi = -\sin^2 \theta \partial_\theta \frac{\beta}{\sin \theta} , \quad \beta_\theta = -\frac{H_2^2}{H_1} \partial_r \frac{\gamma}{H_2} , \quad \partial_r \frac{\gamma}{H_0} = -T^2 \frac{H_1}{H_0^2} \partial_t \frac{\delta}{T} . \tag{92}$$

Two of the most important integrability conditions for these equations are as follows:

$$\dot{T} H'_0 \beta = 0 , \quad [\rho(r) - \varrho(t)] \beta = 0 , \tag{93}$$

where  $\rho$  and  $\varrho$  are given by (7). The first follows from  $\gamma_{t\theta} = \gamma_{\theta t}$  and  $\delta_{r\theta} = \delta_{\theta r}$ , and the second from  $\delta_{t\theta} = \delta_{\theta t}$ . There are also identical conditions with  $\beta$  replaced by  $\alpha$  that can be checked from  $\gamma_{t\varphi} = \gamma_{\varphi t}$ ,  $\delta_{r\varphi} = \delta_{\varphi r}$  and  $\delta_{t\varphi} = \delta_{\varphi t}$ . But noting that  $\beta = 0$  implies  $\alpha = 0$ , we see that the above conditions include the second (see also relations (94)). There are also some other conditions which should be considered in investigating the cases implied by the above conditions. Therefore, we shall present the general solutions in two classes: (A)  $\beta \neq 0$  and (B)  $\beta = 0$ .

The essence of spherical symmetry seems to be encoded in the first equations of (91) and (92), for they do not depend on the metric coefficient functions. We are thus able to start with their general solutions

$$\alpha = a_1 \cos \varphi + a_2 \sin \varphi , \quad \beta = \cos \theta \alpha_\varphi + a_3 \sin \theta , \tag{94}$$

which can easily be verified. Here,  $a_i$  are integration constants.

### A. Solutions for $\beta \neq 0$

For the fulfilment of conditions (93) in the case of nonzero  $\beta$ , two sets of conditions must be distinguished:

$$\begin{aligned} \text{(i)} : \quad & \dot{T} = 0, \quad H'_2 = kH_0H_1, \\ \text{(ii)} : \quad & H'_0 = 0, \quad \dot{T}^2 - T\ddot{T} = -\ell = \frac{H_0^2}{H_1H_2}(H_2P)', \end{aligned}$$

Here  $k$  and  $\ell$  are constants such that  $k \neq 0$ . The well-known maximal symmetric Minkowski and the static form of de Sitter space-times, each having five independent KY 3-forms are obtained among the (i) solutions as special cases. On the other hand, four time dependent forms plus the most well-known form of de Sitter and Robertson-Walker space-times, emerge in the second case. We should emphasize the fact that the former space-time is obtained without any restriction on  $T$ , which is obtained by taking  $H'_0$  zero and by starting in such a way that the last two conditions of (ii) are avoided.

#### 1. KY 3-Forms for the Minkowski and Static Form of de Sitter Space-time

In the (i) case we have, in addition to  $\delta = f(\tau)$ , the following five equations:

$$\begin{aligned} \gamma_\tau &= \frac{H'_0}{H_1}f, \quad \gamma_\theta = -kH_0\beta, \quad \gamma_\varphi = kH_0\sin\theta\alpha, \\ f_\tau &= -\frac{H_0^2}{H_1}\partial_r\frac{\gamma}{H_0}, \quad \beta_\theta = -\frac{H_2^2}{H_1}\partial_r\frac{\gamma}{H_2}. \end{aligned} \tag{95}$$

where  $\tau = t/T$ . Provided that  $m$  is a nonzero constant such that

$$H_2H'_0 - H_0H'_2 = mH_1, \tag{96}$$

the last two equations of (95) imply that  $m\gamma$  must be equal to  $H_2f_\tau - H_0\beta_\theta$ . The first and second equations of (95) in this case require  $km = -1$  and  $f_{\tau\tau} + m_1f = 0$ , where the constant  $m_1$  is defined by

$$H'_0 = km_1H_1H_2. \tag{97}$$

The third equation of (95) is then identically satisfied.

As an alternative approach one can first integrate the  $\gamma_\theta$ -equation with respect to  $\theta$ , and then use it in the  $\gamma_\varphi$ -equation to find  $\gamma = kH_0\beta_\theta + G(\tau, r)$ . The remaining three equations

then give the same solution. As a result, under the (i) conditions the general forms of the coefficient functions for KY 3-form are, in addition to that given by (94), as follows:

$$\gamma = -k(f_\tau H_2 - H_0 \beta_\theta) , \quad \delta = f(\tau) . \quad (98)$$

Depending on the value of  $m_1$ , one can easily write the explicit form of  $f$ . For  $m_1 = 0$  we have, in terms of integration constants  $a, b$ ,

$$f = a + b\tau , \quad H'_2 = kH_0H_1 . \quad (99)$$

and  $H_0^2 = k^{-2}$  from equations (96) and (97). For nonzero  $m_1$ , we have

$$H_0^2 = k_0 + m_1 H_2^2 , \quad H_0 H_1 = k_0 k H'_2 , \quad k_0 k^2 = 1 \quad (100)$$

and  $f$  is as follows ( $k_0, b_1$  and  $b_2$  are integration constants):

$$f = \begin{cases} b_1 \cos \omega_0 \tau + b_2 \sin \omega_0 \tau ; & m_1 = \omega_0^2 > 0 , \\ b_1 \cosh \omega_0 \tau + b_2 \sinh \omega_0 \tau , & m_1 = -\omega_0^2 < 0 , \end{cases} \quad (101)$$

In any case, we have five linearly independent 3-forms.

For  $T = 1$ ,  $H_0 = 1 = H_1$  and  $H_2 = r$  we get, from the equations (94), (97) and (99), 3-forms of Minkowski space-time:

$$\begin{aligned} \omega_1 &= w_1 + \sin \theta \sin \varphi e^{023} , & \omega_2 &= w_2 - \sin \theta \cos \varphi e^{023} , \\ \omega_3 &= \sin \theta e^{013} + \cos \theta e^{023} , \\ \omega_4 &= e^{123} , & \omega_5 &= -r e^{023} + \tau e^{123} , \end{aligned} \quad (102)$$

where the 3-forms  $w_1$  and  $w_2$  are defined, for brevity, as

$$w_1 = \cos \varphi e^{012} - \cos \theta \sin \varphi e^{013} , \quad w_2 = \sin \varphi e^{012} + \cos \theta \cos \varphi e^{013} . \quad (103)$$

Note that  $k = 1$  (hence  $m = -1$ ) and  $m_1 = 0$  for this case. On the other hand, for the values

$$T = 1 , \quad H_2 = r , \quad k_0 = 1 = k \quad (104)$$

we obtain, by virtue of equations (94), (97) and (100), the KY 3-forms

$$\begin{aligned} \omega_1 &= w_1 + H_0 \sin \theta \sin \varphi e^{023} , & \omega_2 &= w_2 - H_0 \sin \theta \cos \varphi e^{023} , \\ \omega_3 &= \sin \theta e^{013} + H_0 \cos \theta e^{023} , & \omega_4 &= r \sinh \tau e^{023} + \cosh \tau e^{123} , \\ \omega_5 &= -r \cosh \tau e^{023} + \sinh \tau e^{123} \end{aligned} \quad (105)$$

for the static form of de Sitter space-time, specified also by  $H_1^2 = 1 + m_1 r^2$  and  $H_0 H_1 = 1$ .

## 2. Solutions for Four Time-Dependent Forms of de Sitter Space-time

From here on, we consider the (ii) conditions. The condition  $H'_0 = 0$  gives  $\gamma_t = 0$  and leaves us with seven equations to be solved. It is easy to integrate the  $\gamma_\theta$ -equation of (91) and then use it in the  $\gamma_\varphi$ -equation to find  $\gamma = H_2 P \beta_\theta + G(r)$ . By substituting this solution into the  $\beta_\theta$ -equation, we find  $G = c H_2$  and hence,  $\gamma = H_2 P \beta_\theta + c H_2$ , provided that  $P' = -H_1/H_2^2$ . Here,  $c$  is an integration constant. The  $\delta_\theta$ -equation can also be integrated with respect to  $\theta$ , and then by substituting the solution into  $\delta_\varphi$ -equation, we find

$$\delta = \dot{T} \frac{H_2}{H_0} \beta_\theta + D(t, r) . \quad (106)$$

The following two equations of (91) and (92) remain to be solved:

$$\delta_r = \dot{T} \frac{H_1}{H_0} \gamma , \quad \frac{H_0}{H_1} \gamma_r = -T^2 \partial_t \frac{\delta}{T} .$$

By substituting the above  $\gamma$  solution and (106) into these equations, we obtain

$$D_r = c \dot{T} \frac{H_1 H_2}{H_0} , \quad T^2 \partial_t \frac{D}{T} = -c H_0 H_2 P , \quad (107)$$

provided that

$$(H_2 P)' = -\ell \frac{H_1 H_2}{H_0^2} , \quad (108)$$

which is just one of the integrability conditions of (ii). Now it is not difficult to see that for nonzero values of  $\ell$ , the general solution for  $D$  is  $D = -(c/\ell) \dot{T} H_0 H_2 P + c_0 T$ , where  $c_0$  is another integration constant. We can now collate the solutions of the case as follows:

$$\begin{aligned} \alpha &= a_1 \cos \varphi + a_2 \sin \varphi , \quad \beta = \cos \theta \alpha_\varphi + a_3 \sin \theta , \\ \gamma &= H_2 P \beta_\theta + c H_2 , \\ \delta &= \dot{T} \frac{H_2}{H_0} \beta_\theta - \frac{c}{\ell} \dot{T} H_0 H_2 P + c_0 T . \end{aligned} \quad (109)$$

Depending on the values of  $\ell$  and other integration constants arising when integrating the equation  $T^2 \partial_t (\dot{T}/T) = \ell$ , four different time regimes were presented in Part I (see relations (72) in I). The above solutions provide five independent KY 3-forms:

$$\begin{aligned} \omega_1 &= \cos \varphi e^{012} - \cos \theta \sin \varphi e^{013} + H_2 \sin \theta \sin \varphi B , \\ \omega_2 &= \sin \varphi e^{012} + \cos \theta \cos \varphi e^{013} - H_2 \sin \theta \cos \varphi B , \\ \omega_3 &= \sin \theta e^{013} + H_2 \cos \theta B , \\ \omega_4 &= H_2 (e^{023} - \frac{H_0}{\ell} \dot{T} P e^{123}) , \\ \omega_5 &= T e^{123} , \end{aligned} \quad (110)$$



where we have defined the 3-form  $B = Pe^{023} + H_0^{-1}\dot{T}e^{123}$ .

### 3. KY 3-Forms of de Sitter Space-Time with Exponential Time Dependence

When  $\ell = 0$ , we have  $\dot{T} = \lambda T$  and  $H_2' = m_0 H_1$  from (108). In such a case, the most general solution of (107) turns out to be

$$D = c \frac{\lambda T}{2m_0 H_0} [H_2^2 + (\frac{m_0 H_0}{\lambda T})^2] + c_1 \frac{1}{T} ,$$

where  $c_1$  is an integration constant. The solutions are then given by (109), with  $\delta$  replaced by  $\delta = \dot{T}(H_2/H_0)\beta_\theta + D$ . From  $P' = -H_1/H_2$ , it follows that  $m_0^2 = 1$ , that is,  $m_0 = \varepsilon = \pm 1$  and we again obtain five linearly independent 3-forms for de Sitter space-time. The first three forms are the same as those given in (110), and the last two are as follows:

$$\omega_4 = H_2 e^{023} + \frac{\lambda T}{2m_0 H_0} [H_2^2 + (\frac{m_0 H_0}{\lambda T})^2] e^{123} , \quad \omega_5 = \frac{1}{T} e^{123} . \quad (111)$$

### 4. KY 3-form of the Robertson-Walker Space-Time

For the solutions obtained so far under (ii) conditions,  $T$  is restricted as a special function of time. It turns out (see also the next section) that the only possible solution in which there are no constraints on  $T$  is

$$\omega = T e^{123} , \quad (112)$$

provided that  $H_0' = 0$ . In particular, there is only one KY 3-form for the Robertson-Walker space-time.

## B. Solutions for $\beta = 0$

For  $\beta = 0$ , the equations (91) and (92) imply that  $\alpha = 0$ ,  $\gamma = H_2 f(t)$  and  $\delta = D(t, r)$ , and the following three equations

$$\begin{aligned} T\dot{f} &= \frac{H_0'}{H_1 H_2} D , \\ \dot{T}f &= \frac{H_0}{H_1 H_2} D_r , \\ \frac{T^2}{f} \partial_t \frac{D}{T} &= -\frac{H_0^2}{H_1} \left( \frac{H_2}{H_0} \right)' , \end{aligned} \quad (113)$$

determine  $f$  and  $D$ . If  $H'_0$  is zero, then  $f$  is a constant, say  $c$ , and we get  $\gamma = cH_2$  and  $\delta = -(c/\ell)\dot{T}H_0H_2P + c_0T$ , which are special cases of solutions (109). Moreover, the special case  $c = 0$  produces the solution (112) for the Robertson-Walker space-time.

For nonzero  $H'_0$ , taking  $D$  from the first equation of (113) and substituting it into the other two equations lead us to two separate equations. Each side of these equations must be constant such that

$$\frac{H_0}{H_1H_2}\left(\frac{H_1H_2}{H'_0}\right)' = m_1, \quad \frac{H_0^2H'_0}{H_1^2H_2}\left(\frac{H_2}{H_0}\right)' = m_2. \quad (114)$$

This leaves us with two simple equations for  $f$ :

$$\frac{\dot{T}f}{T\dot{f}} = m_1, \quad T^2\frac{\ddot{f}}{f} = -m_2. \quad (115)$$

For  $m_1 = 0$ , we have  $\dot{T} = 0$  and

$$f_{\tau\tau} + m_2f = 0, \quad H'_0 = m_3H_1H_2, \quad D = m_3^{-1}f_\tau. \quad (116)$$

where  $m_3$  is a nonzero constant. These provide two independent 3-forms:

$$\omega_1 = \tau H_2 e^{023} + \frac{1}{m_3} e^{123}, \quad \omega_2 = H_2 e^{023},$$

for  $m_2 = 0$  and

$$\begin{aligned} \omega_1 &= H_2 \cosh w_0 \tau e^{023} + \frac{w_0}{m_3} \sinh w_0 \tau e^{123}, \\ \omega_2 &= H_2 \sinh w_0 \tau e^{023} + \frac{w_0}{m_3} \cosh w_0 \tau e^{123}, \end{aligned}$$

for  $m_2 = w_0^2 > 0$ . For nonzero values of  $m_1$ , the first equation of (115) can be easily integrated to yield  $f = c_1 T^{1/m_1}$ , and by substituting this into the second equation, we get  $\ddot{K} = -m_2 K^{1-2m_1}$ , where  $K = T^{1/m_1}$ . As long as this last condition and that given by (114) are satisfied, the general solutions which define only one KY 3-form are as follows:

$$\gamma = c_1 H_2 T^{1/m_1}, \quad \delta = \frac{c_1}{m_1} \dot{T} T^{1/m_1} \frac{H_1 H_2}{H'_0}.$$

## VIII. CONCLUSION

By directly starting from the KY-equation, we have developed a constructive method which makes it possible to generate all KY forms for a large class of spherically symmetric

space-times in a unified and exhaustive way. Our results for the well-known spherically symmetric space-times are quantitatively summarized in Table I of the first paper and their KY two and three forms are computed in this second paper. In particular, we have found an exactly solvable nonlinear time equation for de Sitter type space-times which enables us to generate all of their KY forms in a unified manner. We have also reported solutions in some detail for sufficiently symmetric new cases which fall within the considered class of metrics.

Our results can be used to reach decisive, or at least conclusive statements in analyzing the algebraic structures of KY-forms [6, 9, 10], in specifying of the symmetry algebra and related conserved quantities of the Dirac as well as other equations in spherically symmetric curved backgrounds [11, 12, 13, 14, 15]. Finally, as an application, we indicate an approach for calculating Killing tensors and associated first integrals for the considered class of space-times [16]. As has been mentioned before, to each KY  $(p+1)$ -form  $\omega$ , there corresponds an associated Killing tensor  $K$  that can be defined by  $K(X, Y) = g_p(i_X\omega, i_Y\omega)$ , where  $g_p$  is the compatible metric in the space of  $p$ -forms induced by the space-time metric  $g$ . Then  $i_{\dot{\gamma}}\omega$  is parallel-transported along the affine-parameterized geodesic  $\gamma$  with tangent field  $\dot{\gamma}$ , and  $K(\dot{\gamma}, \dot{\gamma})$  is the associated quadratic first integral. The first statement follows from the fact that the covariant and interior derivatives with respect to the same geodesic tangent field commute and the second statement follows from the fact that the cyclicly permuted sum of  $\nabla_X K(Y, Z)$  vanishes. In particular, Killing tensor fields and associated first integrals for the space-times given in the Table I of I can be computed and used in investigating some integrability problems. Our study on the symmetries of the Dirac equation and related matter, is in progress, and soon will be reported elsewhere [17].

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